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Construction and Performance of Network Codes.

J. de Curtó i Díaz.

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Outline

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Physical-layer Network Coding

Introduction

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Today

Interference is treated as a destructive phenomenon.

Network Coding Introduced the Idea

Intermediate nodes in a network are able to perform operations to the input packets rather than just forwarding them.

Network Coding at the Physical-layer? PNC

When multiple electromagnetic waves come together within the same physical space, they add. This additive mixing of electromagnetic waves is a form of Network Coding, performed by nature. PNC aims to exploit this fact.

Physical-layer Network Coding

Main ideas

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The Source Transmits a Message

$\mathbf{w}_l \in \mathbb{F}_p^k$, where $\mathbb{F}_p$ is a finite field with p elements $\{0, 1, 2 \ldots, p-1\}$ and p is a prime number.

The Relay Decodes a Linear Combination $\mathbf{v}$ of these Messages

$\mathbf{v} = a_1\mathbf{w}_1 \oplus a_2\mathbf{w}_2 \oplus \ldots a_L\mathbf{w}_L$, where a_l are coefficients over the finite field $\mathbb{F}_p$.

The Destination Can Solve For the Original Messages if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{1L} \\ a_{21} & a_{22} & a_{2L} \\ \vdots & \vdots & \vdots \\ a_{M1} & a_{M2} & a_{ML} \end{bmatrix}$$ has rank L.

Physical-layer Network Coding

Example

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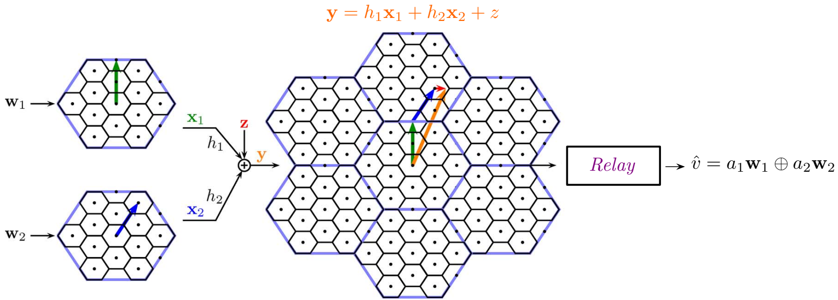

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Lattice Network Codes

Introduction

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What Are We Looking For?

If the waveforms at the transmitter are points of a lattice (that is $\mathbb{Z}$ or $\mathbb{Z}[i]$), then every integer combination of these waveforms is itself a point of the same lattice.

The Algebraic Structure Necessary Is

Given a R-lattice Λ (e.g. $\mathbb{Z}[i]$) and a sublattice Λ' of Λ (e.g. $\pi\mathbb{Z}[i]$), the quotient group Λ/Λ' (e.g. $\frac{\mathbb{Z}[i]}{\pi\mathbb{Z}[i]}$) is a R-module. For a Lattice Network Code, the message space is $\mathrm{W} = \Lambda/\Lambda'$.

Let's See a Bit More of Insight

The R/aR structure, being R a PID and a prime, forms a field. Thus, we will be able to find an isomorphism between $\mathbb{F}_p = \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $\frac{\mathbb{Z}[i]}{\pi\mathbb{Z}[i]}$ if both fields have the same number of elements.

Lattice Network Codes

The Lattice $\mathbb{Z}[i]$

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GAUSSIAN Integers

$\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$.

Prime Factorization Used

If $p \equiv 1 \bmod 4$ then $p = \pi\pi^*$ is a product of two conjugate primes π, π^*.

Example

The prime $p = 5$ satisfies $5 \equiv 1 \bmod 4$, so 5 has two conjugate GAUSSIAN prime factors.
Since $5 = 1^2 + 2^2$, $5 = (1 + 2i)(1 - 2i)$.

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3 C&F

- C&F System: Scalar Case
- C&F System: Vectorial Case

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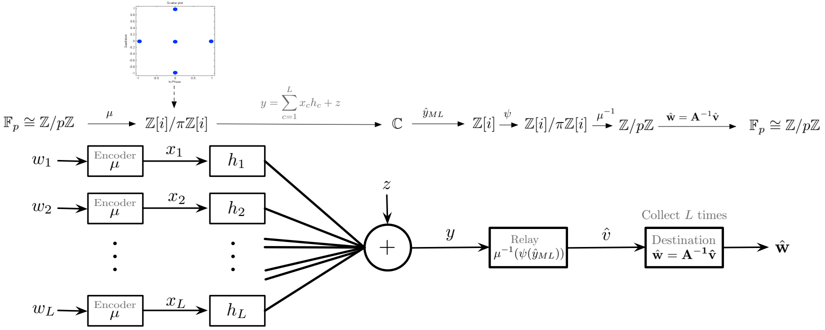
C&F System: Scalar Case

System Model



$$
\begin{aligned}
\mu(w_l) &= w_l \bmod \pi = w_l - \left[\frac{w_l \pi^*}{\pi \pi^*}\right]\pi \\
\mu^{-1}(z) &= z \bmod p = (z^* u \pi + z v \pi^*) \bmod p \\
y &= h_1 x_1 + h_2 x_2 + \ldots + h_L x_L + z \\
\hat{v} &= a_1 w_1 \oplus a_2 w_2 \oplus \ldots \oplus a_L w_L
\end{aligned}
$$

C&F System: Scalar Case

Performance

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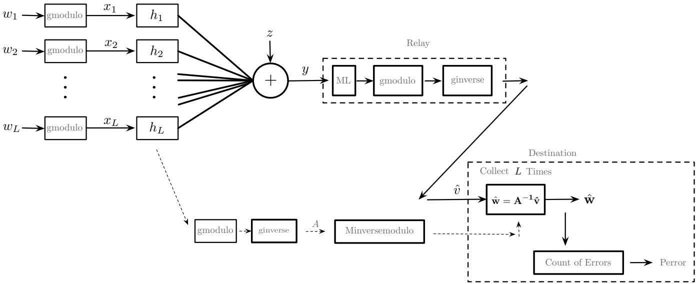

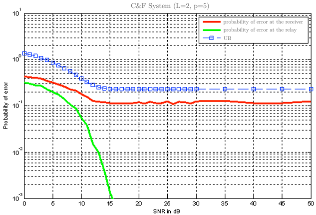

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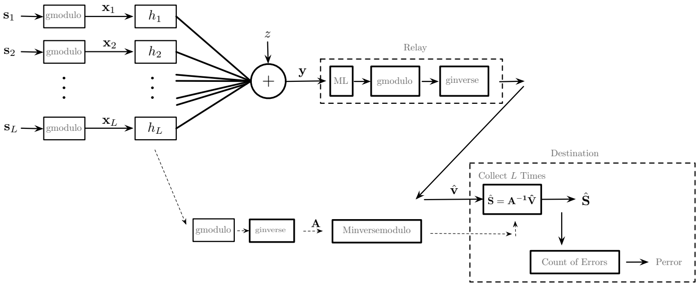
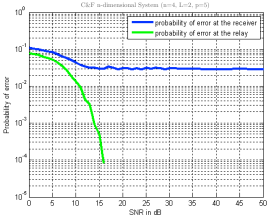
C&F System: Vectorial Case

Performance





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HAMMING q-ary Codes

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Generating Matrix

$$C = uG : u \in \mathbb{F}_p^k.$$

We say that G is systematic if $G = (I_k | -P^T)$.

Parity Check Matrix

$$C = \{v \in \mathbb{F}_p^n : Hv^T = 0\}.$$

If G is systematic, a parity check matrix is $H = (P | I_{n-k})$.

C&F HAMMING q-ary Coded System

Construction

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HAMMING q-ary Code

Given an integer $r \geq 2$, let $n = \frac{q^r-1}{q-1}$. The HAMMING q-ary code is a linear $[n, n-r]$ code in $\mathbb{F}_q^n$, whose parity check matrix H is such that

$$H = (v_1|v_2|\dots|v_n)$$

where $v_1, \dots, v_2 \in F_q^r$ is a list of nonzero vectors satisfying the condition that no two vectors are scalar multiples of each other.

Example

Let $\mathbb{F}^5$ and $r = 2$, $n = \frac{5^2-1}{5-1} = 6$. So, $k = n - r = 4$. A straightforward way to generate a systematic HAMMING q-ary code is generating the matrix P as a $r \times k$ matrix with columns

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

And then generate H and G using $G = (I_k| - P^T)$ and $H = (P|I_{n-k})$.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 0 & 4 & 3 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & 4 & 1 \end{bmatrix}.$$

C&F HAMMING q-ary Coded System

Performance

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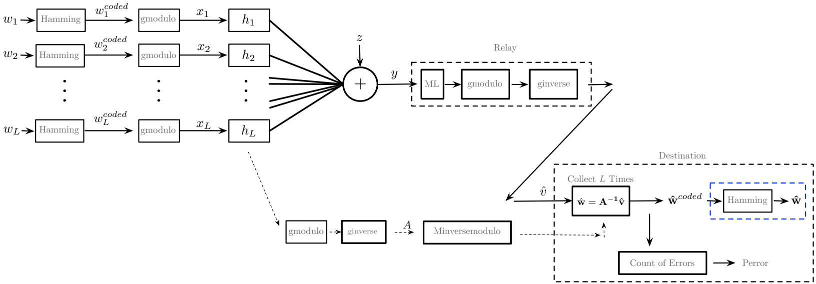

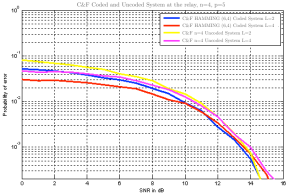

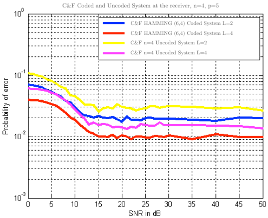

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5 Improvement of the Coefficients
- Improvement of the Coefficients: Improved Matrix **A**
- Improvement of the Coefficients: Optimum Matrix **A**
- Improvement of the Coefficients: Improved Optimum Matrix **A**

Improved Matrix A

Performance

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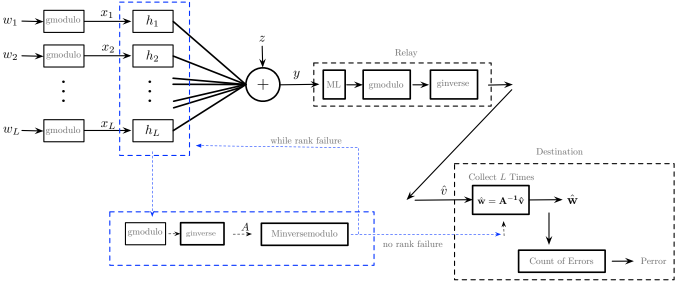

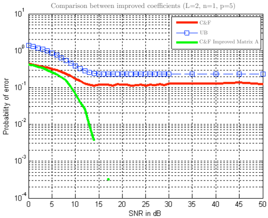

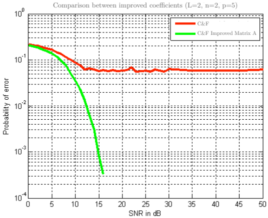

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Optimum Matrix A

Construction

Scalar Factor

$$\beta_{MMSE} = \frac{\mathrm{SNR}\mathbf{h}^T\mathbf{a}_m}{\mathrm{SNR}||h_m||^2 + 1}.$$

Optimum Coefficients

Theorem: For a given vector of coefficients of the channel $\mathbf{h}_m = [h_{m1}, h_{m2}, \ldots, h_{mL}]^T \in \mathbb{R}^L$, the computation rate is maximized by choosing in network coding the vector of coefficients $\mathbf{a}_m \in \mathbb{Z}^L$ as

$$\mathbf{a}_m = \arg \min_{\mathbf{a}_m \in \mathbb{Z}^L, \mathbf{a}_m \neq 0} (a_m^T G_m a_m)$$

where

$$\mathbf{G}_m = \mathbf{I} - \frac{\mathrm{SNR}}{1 + \mathrm{SNR}||\mathbf{h}_m||^2 \mathbf{H}_m}.$$

Optimum Matrix **A**

Construction

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What's Behind this Minimization?

$$\mathbf{a}_m = \arg \min_{\mathbf{a}_m \in \mathbb{Z}^L, \mathbf{a}_m \neq 0} (a_m^T G_m a_m).$$

- CHOLESKY factorization.
- Lattice reduction: LLL algorithm.
- Vector search: SCHNORR EUCHNER method.

Optimum Matrix A

Solving the ILS Problem

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ILS Problem

$$\min_{z\in\mathbb{Z}^n} ||\mathbf{y}-\mathbf{Bz}||^2$$

this problem is analogous to solving

$$\min_{z\in\mathbb{Z}^n} (\mathbf{y}-\mathbf{Bz})^T\mathbf{V}^{-1}(\mathbf{y}-\mathbf{Bz}).$$

One can first compute the CHOLESKY factorization $\mathbf{V}=\mathbf{R}^T\mathbf{R}$, then solve two lower triangular linear systems $R^T\overline{y}=y$ and $R^T\overline{B}=B$.
As our real aim is to solve the SVP problem

$$\min_{z\in\mathbb{Z}^n} (\mathbf{z})^T\mathbf{V}^{-1}(\mathbf{z})$$

we use $B=-I_n$ and $y=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}_n$ and therefore $\overline{B}=R^T\backslash B$ and $\overline{y}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}_n$.
Finally the problem becomes

$$\min_{z\in\mathbb{Z}^n} ||\overline{\mathbf{y}}-\overline{\mathbf{B}}\mathbf{z}||^2.$$

Optimum Matrix A

Performance

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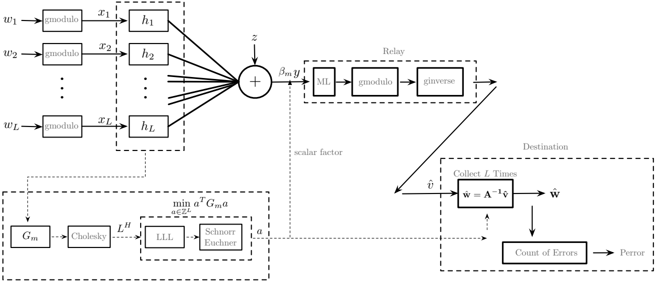

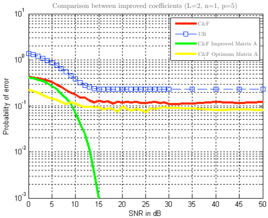

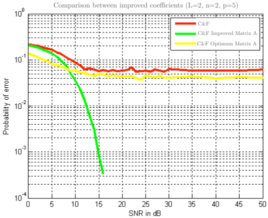

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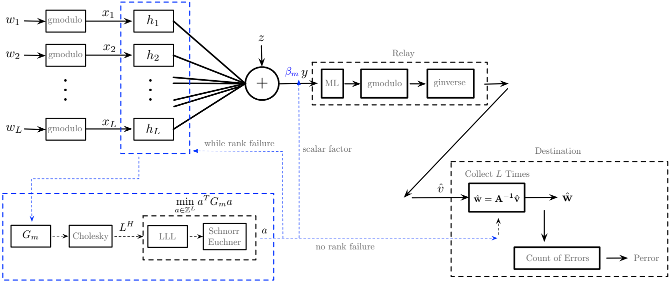
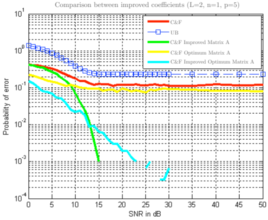
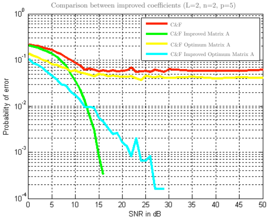
Improved Optimum Matrix A

Performance







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Conclusions

- Mathematical tools. $\surd$
- C&F uncoded system model scalar:
  - MATLAB L-dimensional $\forall p$ implementation using a working $L = 2, p = 5$ code base with given ML detector. $\surd$
- C&F uncoded system model vectorial:
  - MATLAB implementation n-dimensional. $\surd$
- C&F HAMMING q-ary coded system model:
  - MATLAB implementation C&F HAMMING (6,4) coded system $n = 4$. $\surd$
- Improvement of the Coefficients:
  - MATLAB implementation improved matrix **A**. $\surd$
  - MATLAB implementation optimum matrix **A**. $\surd$
  - MATLAB implementation improved optimum matrix **A**. $\surd$
- Implementation of sphere decoder for ML detection:
  - Adapting the code used for optimum matrix **A** as an efficient sphere decoder. $\surd$

Conclusions

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Obtained Results

- We have explained the lattice theory needed.
- We have provided several MATLAB code implementations for C&F system.
- The results of the improvement of the coefficients show:
  - Improved optimum matrix $\mathbf{A}$ works really well for SNR low.
  - Improved matrix $\mathbf{A}$ has a better slope performance for SNR high.

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Thank You

DE CURTÓ I DÍAZ Joaquim.

Thank you.